Math 40960, Topics in Geometry

Problem Set 3, due April 16, 2014

<u>**Note**</u>: Answer the questions as I've written them here; the references to the books are just so you know where they came from. Of course you should show all your work. And as always, if you get any help from any source, in person or online or otherwise, you need to acknowledge it.

1. (a) Find the pure O-sequence generated by the monomials $\{x^5, y^5, z^5, x^2y^2z, xyz^3, x^3yz\}$ in the variables x, y, z.
   (b) Find a set of monomials that generate the pure O-sequence $(1, 4, 7, 5, 2)$.
   (c) Let $\mathbb{P}^2(\mathbb{Z}_5)$ be the classical projective plane associated to the field $\mathbb{Z}_5$. Suppose you choose **only two** of the lines, ℓ_1 and ℓ_2, of $\mathbb{P}^2(\mathbb{Z}_5)$ and assign distinct variables only to the points of these two lines. Assign to ℓ_1 the monomial M_1 given by the product of the variables associated to the points of ℓ_1, and similarly define M_2 using the points of ℓ_2. What is the pure O-sequence that results from the monomials $\{M_1, M_2\}$?

2. Let $\mathbb{A}^2$ be a **finite** affine plane of order n. Recall the following facts about affine planes of order n (some are axioms and some were proved in class). You can use any or all of these facts freely in this problem, and you do not have to prove any of these facts. (I suspect that they will be more helpful for part (c) than for any of the other parts.)

Fact 1: For every two distinct points $A, B \in \mathbb{A}^2$ there is a unique line containing them.
Fact 2: For every line ℓ and every point A there exists a unique line through A and parallel to ℓ.
Fact 3: There exist four points in $\mathbb{A}^2$ such that no three are collinear.
Fact 4: Two lines in $\mathbb{A}^2$ are either parallel or they meet in one point.
Fact 5: Every line contains n points.
Fact 6: Every point lies on $n + 1$ lines.
Fact 7: There are $n + 1$ pencils of parallel lines.
Fact 8: Every pencil consists of n lines, which partition the points of $\mathbb{A}^2$.
Fact 9: $\mathbb{A}^2$ contains n^2 points and $n^2 + n$ lines.

Recall also that in class we proved that there exists a **projective** plane of order n if and only if

$$\left(1, N, N \cdot \binom{n+1}{2}, N \cdot \binom{n+1}{3}, \ldots, N \cdot \binom{n+1}{n-1}, N \cdot \binom{n+1}{n}, N\right)$$

is a pure O-sequence, where $N = n^2 + n + 1$.

In this problem you will be asked to find and prove the analogous statement for affine planes. Here (finally!) is what you have to do for this problem.

(a) Find the analogous pure O-sequence for affine planes. (Don't prove anything yet.)

(b) Prove that $\binom{n^2}{2} = (n^2 + n) \cdot \binom{n}{2}$ for any positive integer $n \geq 2$.

(c) Prove that if there exists an affine plane of order n then your answer to (a) is a pure O-sequence.

(d) Prove that if your answer to (a) is a pure O-sequence then there exists an affine plane of order n.

3. (From Hartshorne, Exercise 7.2) As we said in class, statements that are "obvious" still need to be proven, to avoid the kinds of issues we discussed with Euclid's Elements. The following problems are to be proven from the axioms and/or the theorems and propositions in section 7 of Hartshorne's book, without appealing to intuition.

   (a) Let $\overline{AB}$ be a segment. Show that there do not exist points $C, D \in \overline{AB}$ such that $C * A * D$.

   (b) Conclude from part (a) that the endpoints, A, B of the segment $\overline{AB}$ are uniquely determined by the segment.

4. (This is basically Hartshorne, Exercise 7.4.) Recall Axiom (B2):

   For any two distinct points A and B there exists a point C such that $A * B * C$.

   It would be tempting to try to conclude that this axiom **by itself** means that every line has infinitely many points, since if you start with A_1 and A_2, there exists a point A_3 such that $A_1 * A_2 * A_3$. Then you take A_2 and A_3 and it gives a point A_4 such that $A_2 * A_3 * A_4$, and so on. But if you think about it, there's no reason why you can't re-use A_1 or something similar, unless you have some sort of result or axiom that precludes it.

   So being careful to justify your steps, prove that if all the axioms (I1), (I2), (I3), (B1), (B2), (B3), (B4) hold (and hence all the results from section 7 of Hartshorne hold!), then every line has infinitely many points.

5. (This is basically Hartshorne, Exercise 7.5) The purpose of this exercise is to construct a model of points on a line that satisfy (B1), (B2), (B3) but not (B4). We don't care about the incidence axioms.

   Consider the set of elements of $\mathbb{Z}_5$. Think of $\mathbb{Z}_5$ as being the points on a line (so the line contains exactly five points). For $a, b, c \in \mathbb{Z}_5$, define $a * b * c$ if and only if $b = 3(a + c)$ (remembering that the product is mod 5). In this problem, please label the points using the notation of $\mathbb{Z}_5$, so the points are $\{0, 1, 2, 3, 4\}$ and you just have to define "betweenness" for these five points.

   (a) For any three elements a, b, c of $\mathbb{Z}_5$ for which $a * b * c$ is true, prove that $c * b * a$ is also true. This is axiom B1.

   (b) For each way of choosing three of the five elements, $\{a, b, c\}$, of $\mathbb{Z}_5$ (and there are $\binom{5}{3} = 10$ such choices), say which of those three elements is between the other two, i.e. either $a*b*c$ or $b*a*c$ or $a * c * b$ (taking into account part (a)). This is axiom B3.

   (c) Prove that axiom B2 holds for this model. (You are free to use your work from earlier parts of this problem.) Remember that choosing A and then B is not the same as choosing B and then A.

   (d) Prove directly that this definition of $a * b * c$ does not satisfy line separation (Proposition 7.2 of Hartshorne). (I.e. show **directly** that line separation does not hold, rather than giving an indirect proof.)

   (e) What's the connection between this problem and problem #4?